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A STUDY OF SLANT WEIGHTED TOEPLITZ OPERATORS IN CALKIN ALGEBRA

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Abstract. We describe some structural properties of a k^{th}-order slant weighted Toeplitz operator $U^{\beta}_{k,\phi}$ $(k \geq 2)$ on $L^2(\beta)$ with $\phi \in L^{\infty}(\beta)$. The study of this operator and its counterpart on $H^2(\beta)$ is also extended in reference to the Calkin algebra.

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1. Introduction and Preliminaries

M.C. Ho [7], in the year 1995, introduced the notion of a slant Toeplitz operator on L^2 $(= L^2(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle in the complex plane) with the property that its matrix with respect to the standard orthonormal basis could be obtained by eliminating every alternate row of the matrix of the corresponding Laurent operator. If $\phi = \sum\limits_{n=-\infty}^{\infty} a_n e_n \in L^{\infty}(\mathbb{T})$ with $a_n = \langle \phi, e_n \rangle$, the n^{th}-Fourier coefficient of ϕ, then the matrix of the slant Toeplitz operator induced by ϕ with respect to the standard orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ ($\mathbb{Z}$ denotes the set of integers) of L^2 is given by

$$\begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & a_{-1} & a_{-2} & a_{-3} & \cdots \\ \cdots & a_{1} & a_{0} & a_{-1} & \cdots \\ \cdots & a_{3} & a_{2} & a_{1} & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Equivalently, slant Toeplitz operators on the Hilbert space L^2 are defined as $S_\phi = WM_\phi$ for $\phi \in L^{\infty}(\mathbb{T})$, where $We_{2n} = e_n$, $We_{2n+1} = 0$ for each integer n and M_ϕ is the Laurent operator on L^2 induced by the symbol ϕ. The slant Toeplitz operators are a particular case of Ruelle operators, which play a vital role in ergodic theory. Also, the spectral properties of slant Toeplitz operators find applications in the theory of wavelets (see [6], [11]). For example, L. Villemoes [12] associated the spectral radius of a slant Toeplitz operator with the Besov regularity of solutions of the refinement equation.

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The underlying spaces in these studies are the usual Hardy spaces H^2 and the L^2 spaces. An important direction of study emerged with the work of R.L. Kelley [8], who brought forth the notion of weighted sequence spaces $H^2(\beta)$ and $L^2(\beta)$. These spaces carry their importance because for particular values of β, they coincide with Hardy space, Dirichlet space and Bergman space (see [10]). Shields [10] made a systematic study of the Laurent operator on $L^2(\beta)$ while Lauric [9] studied a particular case of weighted Toeplitz operators on $H^2(\beta)$. These studies and various applications of slant Toeplitz operators motivated the introduction of the classes of slant weighted Toeplitz operators and k^{th}-order slant weighted Toeplitz operators on $L^2(\beta)$ (see [2], [3]).

Further, in 1982, Barría and Halmos [4], characterized the set of essential commutant of the forward unilateral shift operator and this set is referred to as the set of essentially Toeplitz operators. The class of k^{th}-order essentially slant Toeplitz operators, characterized as the solution T of the operator equation $M_zT - TM_{z^k} = K$ for some compact operator K on L^2, was introduced and studied in [1]. The study in this direction motivates us to extend the study of slant weighted Toeplitz operators to the operators which behave essentially in the same manner as these operators do. We introduce the classes of essentially k^{th}-order slant weighted Toeplitz operators and essentially compressed k^{th}-order slant weighted Toeplitz operators on $L^2(\beta)$ and $H^2(\beta)$ respectively and study their properties.

Let us begin with the brief descriptions of the underlying spaces used in the paper. The symbol $\mathbb{C}$ denotes the set of all complex numbers. The space $L^2(\beta)$ consists of all formal Laurent series $f(z) = \sum\limits_{n\in\mathbb{Z}} a_n z^n$, $a_n \in \mathbb{C}$, (whether or not the series converges for any values of z) for which $\|f\|_\beta^2 = \sum\limits_{n\in\mathbb{Z}} |a_n|^2 {\beta_n}^2 < \infty$, where $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is a sequence of positive numbers with $\beta_0 = 1$, $r \le \frac{\beta_n}{\beta_{n+1}} \le 1$ for $n \ge 0$ and $r \le \frac{\beta_n}{\beta_{n-1}} \le 1$ for $n \le 0$, for some $r > 0$. This assumption on β is taken throughout the paper.

$L^2(\beta)$ is a Hilbert space with the norm $\|\cdot\|_\beta$ induced by the inner product

$$\langle f, g\rangle = \sum_{n\in\mathbb{Z}} a_n\, \overline{b}_n {\beta_n}^2,$$

for $f(z) = \sum\limits_{n\in\mathbb{Z}} a_n z^n$, $g(z) = \sum\limits_{n\in\mathbb{Z}} b_n z^n$. The collection $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ forms an orthonormal basis of $L^2(\beta)$.

The collection of all $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ (formal power series) for which $\|f\|_\beta^2 = \sum\limits_{n=0}^{\infty} |a_n|^2 {\beta_n}^2 < \infty$, is denoted by $H^2(\beta)$ and is a subspace of $L^2(\beta)$.

The symbol $L^\infty(\beta)$ denotes the set of formal Laurent series $\phi(z) = \sum\limits_{n\in\mathbb{Z}} a_n z^n$ such that $\phi L^2(\beta) \subseteq L^2(\beta)$ and there exists some $c > 0$ satisfying $\|\phi f\|_\beta \le c\|f\|_\beta$ for each $f \in L^2(\beta)$. For $\phi \in L^\infty(\beta)$, define the norm $\|\phi\|_\infty$ as

$$\|\phi\|_\infty = \inf\{c > 0 : \|\phi f\|_\beta \le c\|f\|_\beta \text{ for each } f \in L^2(\beta)\}.$$

$L^\infty(\beta)$ is a Banach space with respect to $\|\cdot\|_\infty$. $H^\infty(\beta)$ refers to the set of all formal power series ϕ such that $\phi H^2(\beta) \subseteq H^2(\beta)$. We refer to [10], as well as the references therein, for the details of these spaces. Wherever k appears in the paper, it refers to an integer ≥ 2. For a given Hilbert space H, the symbols $\mathfrak{B}(H)$ and $\mathcal{K}(H)$ denote the sets of all bounded linear operators and of all compact operators on H. The symbol $\sigma_p(T)$ the point spectrum of an operator T, while Φ denotes the empty set.

2. Structural properties

A k^{th}-order slant weighted Toeplitz operator $U^{\beta}_{k,\phi}$ (see [3]), induced by $\phi \in L^{\infty}(\beta)$, is an operator on $L^2(\beta)$ defined as $U^{\beta}_{k,\phi} = W_k M^{\beta}_{\phi}$, where W_k is the operator on $L^2(\beta)$ given by

$$W_k e_n(z) = \begin{cases} \frac{\beta_m}{\beta_{km}} e_m(z) & \text{if } n = km \text{ for some } m \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

and M^{β}_{ϕ} is the weighted Laurent operator on $L^2(\beta)$ induced by $\phi \in L^{\infty}(\beta)$. The second-order slant weighted Toeplitz operator $U^{\beta}_{2,\phi}$ is nothing but the slant weighted Toeplitz operator S^{β}_{ϕ} on $L^2(\beta)$ (see [2]).

$U^{\beta}_{k,\phi}$ is a bounded linear operator on $L^2(\beta)$ with $\|U^{\beta}_{k,\phi}\| \leq \|\phi\|_{\infty}$. If $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, then for each integer j, $U^{\beta}_{k,\phi} e_j = \frac{1}{\beta_j} \sum\limits_{n=-\infty}^{\infty} a_{kn-j} \beta_n e_n$. Also, the matrix $[\lambda_{i,j}]_{i,j\in\mathbb{Z}}$ of $U^{\beta}_{k,\phi}$ with respect to the standard orthonormal basis $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ of $L^2(\beta)$ satisfies

$$\lambda_{i+1,j+k} = \frac{\beta_{i+1}}{\beta_i} \frac{\beta_j}{\beta_{j+k}} \lambda_{i,j}, \tag{1}$$

where $\lambda_{i,j} = \langle U^{\beta}_{k,\phi} e_j, e_i \rangle$, for each $i, j \in \mathbb{Z}$. However, we can find bounded operators on $L^2(\beta)$ other than k^{th}-order slant weighted Toeplitz operators having the matrix structure satisfying (1). Following is an example.

Example 1. Consider the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ given by $\beta_n = 2^{|n|}$ for each $n \in \mathbb{Z}$. Define a formal Laurent series $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, where $a_n = \frac{1}{n^2 2^n}$ if $n > 0$ and $a_n = 0$ otherwise. Then, ϕ is analytic in the domain $|z| < 2$ and is bounded as well. Also, $\|M^{\beta}_z\| = \sup \frac{\beta_{n+1}}{\beta_n} = 2$ and $\|M^{\beta^{-1}}_z\|^{-1} = \inf \frac{\beta_{n+1}}{\beta_n} = \frac{1}{2}$. This provides that ϕ is bounded and analytic in the annulus $\|M^{\beta^{-1}}_z\|^{-1} < |z| < \|M^{\beta}_z\|$ and hence on applying [10, Theorem 10'(vii)(b)], we get that $\phi \in L^{\infty}(\beta)$.

Define T on $L^2(\beta)$ as $T = M^{\beta}_{\phi} W_k$. Then T is a bounded operator on $L^2(\beta)$ and for each $n \in \mathbb{Z}$,

$$Te_n = \begin{cases} \frac{1}{\beta_n} \sum\limits_{l\in\mathbb{Z}} a_{l-m} \beta_l e_l & \text{if } n = km \text{ for some } m \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

The matrix of T, $[\alpha_{i,j}]_{i,j\in\mathbb{Z}}$, with respect to the orthonormal basis $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ is given as

$$\alpha_{i,j} = \begin{cases} \frac{\beta_i}{\beta_j} a_{i-m} & \text{if } j = km \text{ for some } m \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

which satisfies the condition (1). Let, if possible, $T = U^{\beta}_{k,\psi}$ for some $\psi(z) = \sum_{n\in\mathbb{Z}} c_n z^n \in L^\infty(\beta)$. Then, $Te_{km} = U^{\beta}_{k,\psi} e_{km}$ for each integer m, which provides that $a_{l-m} = c_{kl-km}$ for each integer l and m. Thus, we conclude that $c_{kl} = a_l$ if $l > 0$ and $c_l = 0$ otherwise. Hence, $\psi(z) = \sum_{n\in\mathbb{Z}} c_n z^n = \sum_{n\geq 1} c_{kn} z^{kn} = \sum_{n\geq 1} a_n z^{kn}$. Now for $e_0 \in L^2(\beta)$, $\psi e_0 = \sum_{n=1}^{\infty} \frac{2^{(k-1)n}}{n^2} e_{kn}$, which does not belong to $L^2(\beta)$ since $||\psi e_0||^2_\beta = \sum_{n=1}^{\infty} (\frac{2^{(k-1)n}}{n^2})^2$, which is not convergent. Hence $\psi \notin L^\infty(\beta)$. This contradicts our assumption and hence T can't be a k^{th}-order slant weighted Toeplitz operator on $L^2(\beta)$.

It is important to note that if the sequence β is such that $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded, then the only bounded operators on $L^2(\beta)$ satisfying the condition (1) are the k^{th}-order slant weighted Toeplitz operators [3].

As we know that in the case when β is a bounded sequence, the space $H^2(\beta)$ coincides with the Hardy space (see [10]), it is important to ascertain if the sequence β, under the assumption of boundedness of $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$, is bounded. We find that β may or may not be bounded. For example, $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ defined as $\beta_n = \sqrt{|n|+1}$ for each $n \in \mathbb{Z}$ is a unbounded sequence while $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ defined as $\beta_0 = 1$ and $\beta_n = 2$ for each $n \in \mathbb{Z}\backslash\{0\}$ is a bounded sequence, both satisfying the requisite condition.

Theorem 2.1. *Let $m \geq 2$ be an integer such that $\{\frac{\beta_{mn}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. The adjoint $U^{\beta*}_{k,\phi}$ of a k^{th}-order slant weighted Toeplitz operator $U^{\beta}_{k,\phi}$ is a m^{th}-order slant weighted Toeplitz operator if and only if $\phi = 0$.*

Proof. Let, if possible, $U^{\beta*}_{k,\phi}$ be a m^{th}-order slant weighted Toeplitz operator. Then,

$$\langle U^{\beta*}_{k,\phi} e_{j+m}, e_{i+1}\rangle = \frac{\beta_{i+1}}{\beta_i}\frac{\beta_j}{\beta_{j+m}}\langle U^{\beta*}_{k,\phi} e_j, e_i\rangle \tag{2}$$

for each $i, j \in \mathbb{Z}$. Let $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$, then $U^{\beta*}_{k,\phi} e_j = \beta_j \sum_{n=-\infty}^{\infty} \overline{a}_{kj-n} \frac{e_n}{\beta_n}$ for each $j \in \mathbb{Z}$. Hence, equation (2) provides that $\overline{a}_{kj-i} = (\frac{\beta_i}{\beta_j}\frac{\beta_{j+m}}{\beta_{i+1}})^2$ $\overline{a}_{kj-i+km-1}$ for each $i, j \in \mathbb{Z}$. Therefore, we obtain that

$$\overline{a}_i = {\beta_{-i}}^2 (\frac{\beta_{mn}}{\beta_{n-i}})^2 \overline{a}_{n(km-1)+i}, \tag{3}$$

$0 \leq i \leq km-2$, for each $n \in \mathbb{Z}$. But $\phi \in L^\infty(\beta) \subseteq L^2(\beta)$, so we have $\sum_{n=-\infty}^{\infty} |a_n|^2 \leq \sum_{n=-\infty}^{\infty} |a_n|^2 {\beta_n}^2 < \infty$. Thus, $a_n \to 0$ as $n \to \infty$. This, together with boundedness of $\{\frac{\beta_{mn}}{\beta_n}\}_{n\in\mathbb{Z}}$ and the fact that $r \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$, for some $r > 0$, helps us to conclude that $a_0 = a_1 = \cdots = a_{km-2} = 0$. As a consequence, equation (3) yields that $a_n = 0$ for each $n \in \mathbb{Z}$. Hence $\phi = 0$.

The converse is straight forward. □

Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be such that $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. Then, some immediate consequences of the above theorem are the following.

Corollary 2.2. *$U^{\beta*}_{k,\phi}$ is a k^{th}-order slant weighted Toeplitz operator if and only if $\phi = 0$.*

Corollary 2.3. *The only self adjoint k^{th}-order slant weighted Toeplitz operator $U^{\beta}_{k,\phi}$ is the zero operator.*

Once we put $k = 2$ in Theorem 2.1, we obtain the following result, which has also been proved in [2].

Corollary 2.4. *The adjoint of a slant weighted Toeplitz operator is a slant weighted Toeplitz operator if and only if $\phi = 0$.*

We recall a result of [5, Theorem 3.2], which states that if the sequence β is such that $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded, then the only compact k^{th}-order slant weighted Toeplitz operator is the zero operator. Since every Hilbert-Schmidt operator is compact, the next theorem is obtained without any extra efforts.

Theorem 2.5. *Let β be such that $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. The k^{th}-order slant weighted Toeplitz operator $U^{\beta}_{k,\phi}$ on $L^2(\beta)$ is Hilbert-Schmidt if and only if $\phi = 0$.*

We would like to add here that once we drop the restriction of boundedness of $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$, we are able to ensure the existence of non-zero k^{th}-order slant weighted Toeplitz operators on $L^2(\beta)$ which are Hilbert-Schmidt and hence compact also. For example, consider the space $L^2(\beta)$ with the sequence β such that $\beta_n = 2^{|n|}$ for each $n \in \mathbb{Z}$. Let $\phi = a_0$, where $0 \neq a_0 \in \mathbb{C}$. Then, $\sum_{j\in\mathbb{Z}} \|U^{\beta}_{k,\phi} e_j\|^2_\beta = |a_0|^2 \sum_{n\in\mathbb{Z}} \frac{\beta_n^2}{\beta_{kn}^2} = |a_0|^2 \sum_{n\in\mathbb{Z}} \frac{1}{2^{2(k-1)|n|}} = |a_0|^2 \Big(\sum_{n=-\infty}^{-1} \frac{1}{2^{-2(k-1)n}} + \sum_{n=0}^{\infty} \frac{1}{2^{2(k-1)n}} \Big) = |a_0|^2 \frac{2^{2(k-1)}+1}{2^{2(k-1)}-1}$ which is finite, so that $U^{\beta}_{k,\phi}$ is a non-zero Hilbert-Schmidt operator on $L^2(\beta)$.

Similarly, we can see that if β is defined as above, each U^{β}_{k,z^i}, $i \in \mathbb{Z}$ is a non-zero Hilbert-Schmidt operator on $L^2(\beta)$.

Next, we investigate some isometric and normal k^{th}-order slant weighted Toeplitz operators on $L^2(\beta)$. We present here below an example of a k^{th}-order slant weighted Toeplitz operator which is neither an isometry nor a normal operator.

Example 2. Consider the sequence β defined as

$$\beta_n = \begin{cases} 1 & \text{if } n \leq 0 \\ n+1 & \text{if } n \geq 1. \end{cases}$$

Then, $r \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$ and $r \leq \frac{\beta_n}{\beta_{n-1}} \leq 1$ for $n \leq 0$, for $r = \frac{1}{2}$. Let $\phi(z) = z^2$. Then $\phi \in L^\infty(\beta)$. Consider the k^{th}-order slant weighted

Toeplitz operator $U^{\beta}_{k,\phi}$ induced by ϕ. Then, the structure of $U^{\beta}_{k,\phi}$ provides that $U^{\beta}_{k,\phi}e_0 = \frac{1}{\beta_0}\sum\limits_{n=-\infty}^{\infty} a_{kn}\beta_n e_n = \begin{cases} 0 & \text{if } k \geq 3 \\ 2e_1 & \text{if } k = 2 \end{cases}$.

Therefore, $||U^{\beta}_{k,\phi}e_0||_\beta \neq ||e_0||_\beta$ and hence this operator is not an isometry. Also, $U^{\beta *}_{k,\phi}e_0 = \beta_0 \sum\limits_{n=-\infty}^{\infty} a_{-n}\frac{e_n}{\beta_n} = a_2\frac{e_{-2}}{\beta_{-2}} = e_{-2}$ and thus the operator under consideration is not normal.

In fact, the k^{th}-order slant weighted Toeplitz operator $U^{\beta}_{k,\phi}$ cannot be an isometry if the inducing symbol $\phi \in L^\infty(\beta)$ is of the type $\phi(z) = \sum\limits_{i=0}^{k-1}(\sum\limits_{n=-\infty}^{\infty} a_{kn+i}z^{kn+i})$, where $a_{kn+i} = 0$ for each n, for some $0 \leq i \leq k-1$.

Remark 1. Let n_0 be a fixed integer and $\phi \in L^\infty(\beta)$ be such that $\phi(z) = \sum\limits_{i=0}^{k-1} a_{kn_0+i}\ z^{kn_0+i}$, where $|a_{kn_0+i}| = 1$ for each $i = 0, 1, \cdots, k-1$. Then, a necessary condition for $U^{\beta}_{k,\phi}$ to be an isometry is that $\beta_{kn} = \beta_{kn-1} = \cdots = \beta_{kn-(k-1)} = \beta_{n+n_0}$ for each $n \in \mathbb{Z}$.

Proposition 2.6. *The k^{th}-order slant weighted Toeplitz operator $U^{\beta}_{k,\phi}$ on $L^2(\beta)$ induced by the Laurent polynomial $\phi(z) = a_{-1}z^{-1}+a_0+a_1z$ is normal if and only if $\phi = 0$.*

Proof. Let $U^{\beta}_{k,\phi}$, where $\phi(z) = a_{-1}z^{-1}+a_0+a_1z$, be normal. Then, $||U^{\beta}_{k,\phi}e_j||_\beta = ||U^{\beta *}_{k,\phi}e_j||_\beta$ for each integer j. In particular, for $j = 0$, this provides that $\sum\limits_{n=-\infty}^{\infty} |a_{kn}|^2{\beta_n}^2 = \sum\limits_{n=-\infty}^{\infty} |\overline{a}_{-n}|^2\frac{1}{\beta_n^2}$. This in turn yields that $a_1 = a_{-1} = 0$. Similarly for $j = 1$, we obtain that $a_0 = 0$. Hence $\phi = 0$. The converse is trivially true. □

We conclude this section with the following information about the k^{th}-order slant weighted Toeplitz operators.

Theorem 2.7. *Let $\phi \in L^\infty(\beta)$ be such that $\phi(z^k) \in L^\infty(\beta)$. Then, $\{0\} \cup \sigma_p(U^{\beta}_{k,\phi}) = \sigma_p(U^{\beta}_{k,\phi(z^k)})$.*

Proof. Let $0 \neq \lambda \in \sigma_p(U^{\beta}_{k,\phi})$. Then there exists $0 \neq f \in L^2(\beta)$ such that $U^{\beta}_{k,\phi}f = \lambda f$. That is, $W_kM^{\beta}_{\phi}f = \lambda f$. Hence, $M^{\beta}_{\phi}f \neq 0$ and

$$\begin{aligned} U^{\beta}_{k,\phi(z^k)}(M^{\beta}_{\phi}f) = W_kM^{\beta}_{\phi(z^k)}M^{\beta}_{\phi}f &= M^{\beta}_{\phi}W_kM^{\beta}_{\phi}f \\ &= M^{\beta}_{\phi}(\lambda f) = \lambda(M^{\beta}_{\phi}f). \end{aligned}$$

Hence, $\lambda \in \sigma_p(U^{\beta}_{k,\phi(z^k)})$. Conversely, let $0 \neq \mu \in \sigma_p(U^{\beta}_{k,\phi(z^k)})$. Then there exists a non-zero vector $g \in L^2(\beta)$ such that $U^{\beta}_{k,\phi(z^k)}g = \mu g$. Hence, $W_kM^{\beta}_{\phi(z^k)}g = M^{\beta}_{\phi}W_kg = \mu g$. Therefore, $W_kg \neq 0$ and $U^{\beta}_{k,\phi}(W_kg) = W_k(U^{\beta}_{k,\phi(z^k)}g) = \mu(W_kg)$. Hence, $\mu \in \sigma_p(U^{\beta}_{k,\phi})$. Since 0 always belongs to $\sigma_p(U^{\beta}_{k,\phi(z^k)})$, the result follows. □

3. Operators in Calkin algebra

If we impose the condition of boundedness of $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$ on β, then the k^{th}-order slant weighted Toeplitz operators on $L^2(\beta)$ are characterized as the solutions X of the operator equation $M_z^\beta X = XM_{z^k}^\beta$ [3]. It is worth mentioning here that dropping the boundedness of $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$ leads the characterization to fail. For, the operator $T = M_\phi^\beta W_k$ on $L^2(\beta)$ (as defined in Example 1), satisfies the operator equation $M_z^\beta T = TM_{z^k}^\beta$, although T is not a k^{th}-order slant weighted Toeplitz operator on $L^2(\beta)$.

We carry forward the study of k^{th}-order slant weighted Toeplitz operators in the Calkin algebra $\mathfrak{B}(L^2(\beta))/\mathcal{K}(L^2(\beta))$. Henceforth, we assume that the sequence β is such that $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. Let us define such operators formally.

Definition: A bounded linear operator X on $L^2(\beta)$ is said to be an essentially k^{th}-order slant weighted Toeplitz operator if it satisfies the operator equation

$$M_z^\beta X - XM_{z^k}^\beta = K,$$

for some compact operator K on $L^2(\beta)$.

We denote the set of all essentially k^{th}-order slant weighted Toeplitz operators on $L^2(\beta)$ by k-$ESWTO(L^2(\beta))$. The following properties are immediate from the definition.

Proposition 3.1. *k-$ESWTO(L^2(\beta)) \cap \mathcal{K}(L^2(\beta)) = \mathcal{K}(L^2(\beta))$.*

Proposition 3.2. *k-$ESWTO(L^2(\beta))$ is a norm-closed vector subspace of $\mathfrak{B}(L^2(\beta))$.*

Remark 2. Since the zero operator on $L^2(\beta)$ is a compact operator, every k^{th}-order slant weighted Toeplitz operator on $L^2(\beta)$ belongs trivially to the set k-$ESWTO(L^2(\beta))$. In fact, if T is any compact perturbation of a k^{th}-order slant weighted Toeplitz operator on $L^2(\beta)$, then $T \in$ k-$ESWTO(L^2(\beta))$.

Theorem 3.3. *Let k_1 and k_2 (both ≥ 2) be two integers. If $A \in k_1$-$ESWTO(L^2(\beta))$ and $B \in k_2$-$ESWTO(L^2(\beta))$, then $AB \in k_1k_2$-$ESWTO(L^2(\beta))$.*

Proof. Let $A \in k_1$-$ESWTO(L^2(\beta))$ and $B \in k_2$-$ESWTO(L^2(\beta))$. Then, $M_z^\beta A - AM_{z^{k_1}}^\beta \in \mathcal{K}(L^2(\beta))$ and $M_z^\beta B - BM_{z^{k_2}}^\beta \in \mathcal{K}(L^2(\beta))$. A simple computation shows that

$$\begin{aligned} M_z^\beta(AB) - (AB)M_{z^{k_1k_2}}^\beta &= (M_z^\beta A)B - A(BM_{z^{k_1k_2}}^\beta) \\ &= (M_z^\beta A)B - A(B(M_{z^{k_2}}^\beta)^{k_1}) \\ &= AM_{z^{k_1}}^\beta B - AM_z^\beta(B(M_{z^{k_2}}^\beta)^{k_1-1}) + K_1 \end{aligned}$$

$$
\begin{aligned}
&= AM^\beta_{z^{k_1}}B - AM^\beta_{z^2}(B(M^\beta_{z^{k_2}})^{k_1-2}) + K_2 \\
&\vdots \\
&= AM^\beta_{z^{k_1}}B - AM^\beta_{z^{k_1}}B + K_{k_1} \\
&= K_{k_1},
\end{aligned}
$$

where $K_i \in \mathcal{K}(L^2(\beta))$ for each $1 \leq i \leq k_1$. This yields that $AB \in k_1k_2$-$ESWTO(L^2(\beta))$. □

Now we find a condition for the product of two essentially k^{th}-order slant weighted Toeplitz operators to be an essentially k^{th}-order slant weighted Toeplitz operator.

Theorem 3.4. *A necessary and sufficient condition for the product of two essentially k^{th}-order slant weighted Toeplitz operators A and B to be an essentially k^{th}-order slant weighted Toeplitz operator is that $A(M^\beta_{z^k} - M^\beta_z)B$ is a compact operator on $L^2(\beta)$.*

Proof. Let $A, B \in k$-$ESWTO(L^2(\beta))$. Suppose $M^\beta_z A - AM^\beta_{z^k} = K_1$ and $M^\beta_z B - BM^\beta_{z^k} = K_2$, where $K_1, K_2 \in \mathcal{K}(L^2(\beta))$. A simple computation shows that

$$
\begin{aligned}
M^\beta_z(AB) - (AB)M^\beta_{z^k} &= (M^\beta_z A)B - A(BM^\beta_{z^k}) \\
&= (AM^\beta_{z^k} + K_1)B - A(M^\beta_z B - K_2) \\
&= A(M^\beta_{z^k} - M^\beta_z)B + K_3,
\end{aligned}
$$

where $K_3 = K_1B + AK_2 \in \mathcal{K}(L^2(\beta))$. This provides that $AB \in$ k-$ESWTO$ $(L^2(\beta))$ if and only if $A(M^\beta_{z^k} - M^\beta_z)B \in \mathcal{K}(L^2(\beta))$. □

To check if the set k-$ESWTO(L^2(\beta))$ forms an algebra or not, we raise the question whether k-$ESWTO(L^2(\beta))$ contains two non compact operators A and B such that $A(M^\beta_{z^k} - M^\beta_z)B$ is a non compact operator. The answer is in positive as is justified by the following example.

Example 3. Let A be an operator on $L^2(\beta)$ defined as

$$
Ae_n = \begin{cases} e_1 & \text{if } n = 0 \\ \frac{\beta_m}{\beta_{km-1}} e_m & \text{if n=km-1 for some m} \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases},
$$

where $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ is the standard orthonormal basis of $L^2(\beta)$.

If K on $L^2(\beta)$ is defined as $Ke_n = e_1$, if $n = 0$ and $Ke_n = 0$ otherwise, then it is easy to see that $A = W_k M^\beta_z + K$. Hence,

$$
\begin{aligned}
M^\beta_z A - AM^\beta_{z^k} &= M^\beta_z(W_k M^\beta_z + K) - (W_k M^\beta_z + K)M^\beta_{z^k} \\
&= (M^\beta_z W_k M^\beta_z - W_k M^\beta_z M^\beta_{z^k}) + K_1 \\
&= (W_k M^\beta_{z^k} M^\beta_z - W_k M^\beta_z M^\beta_{z^k}) + K_1 \\
&= K_1,
\end{aligned}
$$

where $K_1 = M_z^\beta K - KM_{z^k}^\beta \in \mathcal{K}(L^2(\beta))$. Thus A is an essentially k^{th}-order slant weighted Toeplitz operator on $L^2(\beta)$. Also, A is not compact. Let $B = A$. Then, A and B are both non-compact operators belonging to the set k-$ESWTO(L^2(\beta))$. If we assume that $A(M_{z^k}^\beta - M_z^\beta)B$ is compact, then this implies that the operator $W_k M_z^\beta (M_{z^k}^\beta - M_z^\beta) W_k M_z^\beta$ is compact. However, $\left(W_k M_z^\beta (M_{z^k}^\beta - M_z^\beta) W_k M_z^\beta\right) e_n =$

$$\begin{cases} \frac{\beta_{p+1}}{\beta_n} e_{p+1} & \text{if } n = k^2p - k - 1 \text{ for some p} \in \mathbb{Z} \\ -\frac{\beta_p}{\beta_n} e_p & \text{if } n = k^2p - 2k - 1 \text{ for some p} \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Since $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded, there exists a real number $M > 0$ such that $\frac{\beta_{kn}}{\beta_n} \leq M$ for each integer n. Also, $\frac{\beta_n}{\beta n+1} \leq 1$ for each $n \geq 0$. Therefore, $\frac{\beta_n}{\beta_{kn-1}} \geq \frac{1}{M}$ and $\frac{\beta_n}{\beta_{kn-2}} \geq \frac{1}{M}$ for each $n \geq 1$.

Now, the sequence $\{e_{k^2p-2k-1}\}$ converges weakly to 0 as $p \to \infty$. But $\frac{\beta_p}{\beta_{k^2p-2k-1}} = \frac{\beta_{kp-2}}{\beta_{k(kp-2)-1}} \frac{\beta_p}{\beta_{kp-2}} \geq \frac{1}{M^2}$ for each $p \geq 2$. Hence, $||(W_k M_z^\beta (M_{z^k}^\beta - M_z^\beta) W_k M_z^\beta) e_{k^2p-2k-1}||_\beta \not\to 0$ as $p \to \infty$, which contradicts the compactness of this operator.

The following theorem provides a sufficient condition for the product of any two bounded operators on $L^2(\beta)$ to be an essentially k^{th}-order slant weighted Toeplitz operator.

Theorem 3.5. *Let $A, B \in \mathfrak{B}(L^2(\beta))$, then $AB \in$ k-$ESWTO(L^2(\beta))$ if either of the following conditions holds.*

(a) *A is in essential commutant of M_z^β and $B \in$ k-$ESWTO(L^2(\beta))$.*
(b) *$A \in$ k-$ESWTO(L^2(\beta))$ and B is in essential commutant of $M_{z^k}^\beta$.*

Proof. Let $A, B \in \mathfrak{B}(L^2(\beta))$ such that $M_z^\beta A - AM_z^\beta = K_1$ and $M_z^\beta B - BM_{z^k}^\beta = K_2$, where $K_1, K_2 \in \mathcal{K}(L^2(\beta))$. In this case, $M_z^\beta(AB) - (AB)M_{z^k}^\beta = (AM_z^\beta + K_1)B - A(M_z^\beta B - K_2) = K_1 B + AK_2$. Hence, $AB \in$ k-$ESWTO(L^2(\beta)$). Similarly, we can prove the result when condition (b) holds. □

Our next result follows readily using the above theorem.

Proposition 3.6. *Let $\phi \in L^\infty(\beta)$ and $A \in$ k-$ESWTO(L^2(\beta))$. Then, $M_\phi^\beta A$ and AM_ϕ^β both belong to k-$ESWTO(L^2(\beta))$.*

Remark 3. The set k-$ESWTO(L^2(\beta))$ is not a self adjoint set, as can be seen with the help of operator $A = W_k M_z^\beta + K$ (as defined in Example 3). It was proved that $A \in$ k-$ESWTO(L^2(\beta))$. However, if we assume that $A^* \in$ k-$ESWTO(L^2(\beta))$, then $M_z^\beta A^* - A^* M_{z^k}^\beta$ is a compact operator on $L^2(\beta)$. Now,

$$\begin{aligned} M_z^\beta A^* - A^* M_{z^k}^\beta &= M_z^\beta (M_z^{\beta *} W_k^* + K^*) - (M_z^{\beta *} W_k^* + K^*) M_{z^k}^\beta \\ &= (M_z^\beta M_z^{\beta *} W_k^* - M_z^{\beta *} W_k^* M_{z^k}^\beta) + (M_z^\beta K^* - K^* M_{z^k}^\beta) \\ &= T_1 + K_1, \end{aligned}$$

where $T_1 = (M_z^\beta M_z^{\beta *} W_k^* - M_z^{\beta *} W_k^* M_{z^k}^\beta)$ and $K_1 = M_z^\beta K^* - K^* M_{z^k}^\beta \in \mathcal{K}(L^2(\beta))$. Also, $T_1 e_n = (\frac{\beta_n \beta_{kn}}{\beta_{kn-1}^2}) e_{kn} - (\frac{\beta_{n+k}^2}{\beta_n \beta_{k(k+n)-1}}) e_{k(k+n)-1}$. Now, if T_1 is compact, the operator $W_k T_1$ is also compact. But $W_k T_1 e_n = (\frac{\beta_n}{\beta_{kn-1}})^2 e_n$ and hence $||W_k T_1 e_n||_\beta = (\frac{\beta_n}{\beta_{kn-1}})^2 \nrightarrow 0$ as $n \to \infty$. This leads to a contradiction to our assumption.

Towards the end, we focus our attention towards the study of the compression of k^{th}-order slant weighted Toeplitz operators in the Calkin algebra $\mathfrak{B}(H^2(\beta))/\mathcal{K}(H^2(\beta))$. Let V^β denote the shift operator on $H^2(\beta)$ such that for each $j \geq 0$, $V^\beta e_j = \frac{\beta_j}{\beta_{j+1}} e_{j+1}$ and $T_{z^k}^\beta$ denotes the weighted Toeplitz operator on $H^2(\beta)$ induced by z^k.

The compression of X of a k^{th}-order slant weighted Toeplitz operator on $L^2(\beta)$ to $H^2(\beta)$ satisfies the operator equation $X = V^{\beta *} X T_{z^k}^\beta$. We consider the operators satisfying $X - V^{\beta *} X T_{z^k}^\beta \in \mathcal{K}(H^2(\beta))$ and call these as essentially compressed k^{th}-order slant weighted Toeplitz operators.

Also, since $T_z^\beta V^{\beta *} e_n = \begin{cases} 0 & \text{if } n = 0 \\ e_n & \text{if } n \geq 1 \end{cases}$, therefore $T_z^\beta V^{\beta *} = I \ (mod\, \mathcal{K}(H^2(\beta)))$, where I denotes the identity operator on $H^2(\beta)$. Hence, we can equivalently define essentially compressed k^{th}-order slant weighted Toeplitz operators as those satisfying $T_z^\beta X - X T_{z^k}^\beta \in \mathcal{K}(H^2(\beta))$.

We denote the set of all essentially compressed k^{th}-order slant weighted Toeplitz operators on $H^2(\beta)$ by k-$ESWTO(H^2(\beta))$. It is trivial to see that $\mathcal{K}(H^2(\beta)) \subseteq$ k-$ESWTO(H^2(\beta))$. Also, the compression of every k^{th}-order slant weighted Toeplitz operator on $L^2(\beta)$ to $H^2(\beta)$ belongs to k-$ESWTO(H^2(\beta))$.

For $\phi \in L^\infty(\beta)$, let $V_{k,\phi}^\beta$ denote the compression of a k^{th}-order slant weighted Toeplitz operator $U_{k,\phi}^\beta$ to $H^2(\beta)$. Then, $T_z^\beta W_k|_{H^2(\beta)} = V_{k,z^k}^\beta = W_k T_{z^k}^\beta$.

Using this observation, it is easy to see that if T is an operator on $H^2(\beta)$ defined as $T = W_k T_z^\beta + K$, where K is defined on $H^2(\beta)$ as $K e_0 = e_1$ and $K e_n = 0$ if $n \geq 1$, then T is a non compact operator satisfying $T_z^\beta T - T T_{z^k}^\beta \in \mathcal{K}(H^2(\beta))$. Further, along the lines of computations made in Example 3 and Remark 3, we find that $T^2, T^* \notin$ k-$ESWTO(H^2(\beta))$.

With this, we can conclude that k-$ESWTO(H^2(\beta))$ is a proper superset of $\mathcal{K}(H^2(\beta))$ and is neither a self-adjoint set nor an algebra.

Following similar techniques as used in the case of essentially k^{th}-order slant weighted Toeplitz operators on $L^2(\beta)$, we obtain the following information about k-$ESWTO(H^2(\beta))$.

(a) The set k-$ESWTO(H^2(\beta))$ is a norm-closed vector subspace of $\mathfrak{B}(H^2(\beta))$.
(b) Let k_1, k_2 be integers, both ≥ 2. If $A \in k_1$-$ESWTO(H^2(\beta))$ and $B \in k_2$-$ESWTO(H^2(\beta))$, then $AB \in k_1 k_2$-$ESWTO(H^2(\beta))$.
(c) A necessary and sufficient condition for the product of two essentially compressed k^{th}-order slant weighted Toeplitz operators A and B

to be an essentially compressed k^{th}-order slant weighted Toeplitz operator is that $A(T^{\beta}_{z^k} - T^{\beta}_{z})B$ is a compact operator on $H^2(\beta)$.

(d) Let $A, B \in \mathfrak{B}(H^2(\beta))$, then $AB \in$ k-$ESWTO(H^2(\beta))$ if either of the following conditions holds.
  (a) A belongs to the essential commutant of T^{β}_{z} and $B \in$ k-$ESWTO$ $(H^2(\beta))$.
  (b) $A \in$ k-$ESWTO(H^2(\beta))$ and B belongs to the essential commutant of $T^{\beta}_{z^k}$.

Next, we try to determine if an essentially compressed k^{th}-order slant weighted Toeplitz operator on $H^2(\beta)$ can be an invertible operator on $H^2(\beta)$. The following theorem helps us.

Theorem 3.7. *Let $\mathcal{F}(H^2(\beta))$ denotes the set of all Fredholm operators on $H^2(\beta)$. Then, k-$ESWTO(H^2(\beta)) \cap \mathcal{F}(H^2(\beta)) = \Phi$*

Proof. Let $A \in$ k-$ESWTO(H^2(\beta))$ be a Fredholm operator of index n. Then, $T^{\beta}_{z}A = AT^{\beta}_{z^k} + K$, for some compact operator K on $H^2(\beta)$. The index of the operator $T^{\beta}_{z}A$ is n-1, while the index of $AT^{\beta}_{z^k} + K$ is n-k. This implies that $k = 1$ which is absurd. Hence k-$ESWTO(H^2(\beta))$ contains no Fredholm operator. □

Since every invertible operator is a Fredholm operator, the above theorem helps us to conclude that k-$ESWTO(H^2(\beta))$ doesn't contain any invertible operator on $H^2(\beta)$.

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